数理解析研究所講究録 1365 巻 2004 年 10-13

An observation of approximate saddle points

塗矢 哲也 (TETSUYA NURIYA)
島根大学総合理工学研究科 数理・情報システム学専攻
黒岩 大史 (DAISHI KUROIWA)
島根大学総合理工学部 数理・情報システム学科

Abstract

The saddle point is a fundamental concept and in mathematics, economics, and many fields of science. Especially it plays very important roles in game theory, equilibrium theory, and mathematical programming. However, we know that usual theorems of the existence of saddle points, are required conditions with respect to compactness. In this paper we define a notion of approximated saddle points and observe existence of them without compactness.

1 Introduction and Preliminary

Let X and Y be complete metric spaces, f be a function from $X \times Y$ to $\mathbb{R}$. If $(x_0, y_0) \in X \times Y$ is a saddle point of f if for all $(x, y) \in X \times Y$,

$$f(x_0, y) \leq f(x_0, y_0) \leq f(x, y_0).$$

This is a fundamental concept in many fields of science, and it plays important roles in, especially, game theory, equilibrium theory, and mathematical programming. We know the following existence theorem, see [1].

Theorem 1.1 Let X_0 and Y_0 be compact convex subsets of topological vector spaces, and f be a real-valued function on $X_0 \times Y_0$. Assume that $f(\cdot, y)$ is lower semicontinuous quasiconvex for each $y \in Y_0$ and $f(x, \cdot)$ is upper semicontinuous quasiconcave for each $x \in X_0$. Then, there exists a saddle point of f.

However, we know examples in which functions do not have any saddle points when its domain is not compact.

Example 1.1 $f : [1, \infty) \times \mathbb{R} \to \mathbb{R}$ defined by

$$f(x, y) = \left(\frac{1}{x} - 1\right)(y^2 + 1),$$

then there does not exist any saddle points of f.

In this paper, we define a notion of approximated saddle points and observe existence of them without assumption of compactness. To the purpose, we start to remember usual approximation ideas for a minimization problem in the next chapter.

2 Approximate saddle points like Ekeland's method

Let (Z, d) be a metric space. See the following minimization problem (P):

$$\begin{array}{lll} \text{(P)} & \text{minimize} & g(z) \\ & \text{subject to} & z \in Z \end{array}$$

For this problem, we have two approximation ideas: for arbitrary $\varepsilon > 0$,

- $z_0 \in Z$ is (typical) ε-approximate if

$$g(z_0) \leq g(x) + \varepsilon, \quad \forall z \in Z$$

- $z_0 \in Z$ is Ekeland's ε-approximate if

$$g(z_0) \leq g(z) + \varepsilon d(z, z_0), \quad \forall z \in Z.$$

Remember the following Ekeland's theorem; the theorem requires completeness of the metric, but does not require any compactness, see [2].

Theorem 2.1 Let $g : Z \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous, and assume that it is bounded from below. If metric d is complete, then for each $\varepsilon > 0$, there exists $z_0 \in Z$ such that

$$g(z_0) \leq g(z) + \varepsilon d(z, z_0), \quad \forall z \in Z$$

Under the theorem assumptions, function $g(z) + \varepsilon d(z, z_0)$ attains its minimum at z_0, that is, there exists Ekeland's approximate. Motivated the theorem, we define the following approximate saddle point notion.

Definition 2.1 Let $\varepsilon > 0$. $(x_0, y_0) \in X \times Y$ is said to be an Ekeland's ε-approximate saddle point of f if for all $(x, y) \in X \times Y$, two inequalities

$$f(x_0, y) - \varepsilon d(y_0, y) \leq f(x_0, y_0) \quad \text{and} \quad f(x_0, y_0) \leq f(x, y_0) + \varepsilon d(x_0, x)$$

are satisfied.

Remark 2.1 Obviously, if (x_0, y_0) is a saddle point of f, then it is an Ekeland's ε-approximate saddle point of f. Conversely, if (x_0, y_0) is an Ekeland's ε-approximate saddle point of f, then it is a saddle point of the following modified function f_ε:

$$f_\varepsilon(x, y) = f(x, y) + \varepsilon d(x_0, x) - \varepsilon d(y_0, y).$$

Example 2.1 Consider the same function f of Example 1.1, see

$$f(x, y) = \left(\frac{1}{x} - 1\right)(y^2 + 1),$$

then each element of the following set is an Ekeland's ε-approximate saddle point:

$$\left\{ (x, y) \,\middle|\, x^2 - \frac{y^2}{\varepsilon} \geq \frac{1}{\varepsilon},\ |y| \leq \frac{\varepsilon}{2} \right\}.$$

Let $\varepsilon = \frac{1}{4}$. For modified function $f_\varepsilon : [1, \infty] \times \mathbb{R} \to \mathbb{R}$ defined by

$$f_\varepsilon(x, y) = \left(\frac{1}{x} - 1\right)(y^2 + 1) + \varepsilon|x - 4| - \varepsilon|y|$$

has the exact minimax point $(4, 0)$.

3 Existence of Ekeland's approximate saddle points

In this section, we show existence results for our approximate saddle point.

Theorem 3.1 If function f is written by

$$f(x, y) = g(x) - h(y), \quad \forall (x, y) \in X \times Y,$$

where $g : X \to \mathbb{R}$ is lower semicontinuous with bounded from below, and $h : Y \to \mathbb{R}$ is upper semicontinuous with bounded from above. Then for each $\varepsilon > 0$, there exists an Ekeland's ε-approximate saddle point of f.

Theorem 3.2 If function f is written by

$$f(x, y) = g(x)h(y), \quad \forall (x, y) \in X \times Y,$$

where $g : X \to (0, \infty)$ is lower semicontinuous, and $h : Y \to (0, \infty)$ is upper semicontinuous and bounded from above. Then for each $\varepsilon > 0$, there exists an Ekeland's ε-approximate saddle point of f.

The condition of f in Theorem 3.2 is replaced by fractional type as follows:

Corollary 3.1 If function f is written by

$$f(x,y) = g(x)/h(y), \quad \forall(x,y) \in X \times Y,$$

where $g : X \to (0,\infty)$ is lower semicontinuous, and $h : Y \to [c,\infty)$ is lower semicontinuous and c is a positive number. Then for each $\varepsilon > 0$, there exists an Ekeland's ε-approximate saddle point of f.

Theorem 3.3 Assume that f has an Ekeland's ε-approximate saddle point for each $\varepsilon > 0$. If a function $p : X \times Y \to \mathbb{R}$ satisfies η-Lipschitz condition on metric space $(X \times Y, \delta)$ where $\delta((x,y),(x',y')) = d(x,x') + d(y,y')$, and $\eta < \varepsilon$ for given $\varepsilon > 0$, then there exists an Ekeland's ε-approximate saddle point of $f + p$.

Corollary 3.2 If function f is written by

$$f(x,y) = g(x) - h(y) + p(x,y), \quad \forall(x,y) \in X \times Y,$$

where g and h satisfy the same condition in Theorem 3.1, and p satisfies the same condition in Theorem 3.3 for given $\varepsilon > 0$, then there exists an Ekeland's ε-approximate saddle point of f.

By using Theorem 3.3, we can derive similar results concerned with Theorem 3.2 and Corollary 3.1, respectively.

References

[1] M. Sion, On general minimax theorems, *Pacific J. Math.*, **8** (1958) 171–176.

[2] I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974) 324–353.

[3] T. Nuriya and D. Kuroiwa, On the variational principle of minimax functions, preprint.